

# SOME 4-POINT HURWITZ NUMBERS IN POSITIVE CHARACTERISTIC

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**ABSTRACT.** In this paper, we compute the number of covers of curves with given branch behavior in characteristic  $p$  for one class of examples with four branch points and degree  $p$ . Our techniques involve related computations in the case of three branch points, and allow us to conclude in many cases that for a particular choice of degeneration, all the covers we consider degenerate to separable (admissible) covers. Starting from a good understanding of the complex case, the proof is centered on the theory of stable reduction of Galois covers.

## 1. INTRODUCTION

This paper considers the question of determining the number of covers between genus-0 curves with fixed ramification in positive characteristic. More concretely, we consider covers  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  branched at  $r$  ordered points  $Q_1, \dots, Q_r$  of fixed *ramification type*  $(d; C_1, \dots, C_r)$ , where  $d$  is the degree of  $f$  and  $C_i = e_1(i) \cdots e_{s_i}(i)$  is a conjugacy class in  $S_d$ . This notation indicates that there are  $s_i$  ramification points in the fiber  $f^{-1}(Q_i)$ , with ramification indices  $e_j(i)$ . The *Hurwitz number*  $h(d; C_1, \dots, C_r)$  is the number of covers of fixed ramification type over  $\mathbb{C}$ , up to isomorphism. This number does not depend on the position of the branch points. If  $p$  is a prime not dividing any of the ramification indices  $e_j(i)$ , the  *$p$ -Hurwitz number*  $h_p(d; C_1, \dots, C_r)$  is the number of covers of fixed ramification type whose branch points are generic over an algebraically closed field  $k$  of characteristic  $p$ . The genericity hypothesis is necessary because in positive characteristic the number of covers often depends on the position of the branch points.

The only general result on  $p$ -Hurwitz numbers is that they are always less than or equal to the Hurwitz number, with equality when the degree of the Galois closure is prime to  $p$ . This is because every tame cover in characteristic  $p$  can be lifted to characteristic 0, and in the prime-to- $p$  case, every cover in characteristic 0 specializes to a cover in characteristic  $p$  with the same ramification type (see Corollaire 2.12 of Exposé XIII in [9]). We say a cover has *good reduction* when such a specialization exists. However, in the general case, some covers in characteristic 0 specialize to inseparable covers in characteristic  $p$ ; these covers are said to have *bad reduction*. Thus, the difference  $h(d; C_1, \dots, C_r) - h_p(d; C_1, \dots, C_r)$  is the number of covers in characteristic 0 with generic branch points and bad reduction. In [14] and [15], the value  $h_p(d; e_1, e_2, e_3)$  is computed for genus 0 covers and any  $e_i$  prime to  $p$  using linear series techniques. In this paper, we treat the considerably more difficult case of genus-0 covers of type  $(p; e_1, e_2, e_3, e_4)$ . Our main result is the following.

**Theorem 1.1.** *Given  $e_1, \dots, e_4$  all less than  $p$ , with  $\sum_i e_i = 2p + 2$ , we have*

$$h_p(p; e_1, e_2, e_3, e_4) = h(p; e_1, e_2, e_3, e_4) - p.$$

An important auxiliary result is the computation of the  $p$ -Hurwitz number  $h_p(p; e_1-e_2, e_3, e_4)$ .

**Theorem 1.2.** *Given odd integers  $e_1, e_2, e_3, e_4 < p$ , with  $e_1 + e_2 \leq p$  and  $\sum_i e_i = 2p + 2$ , we have that*

$$h_p(p; e_1-e_2, e_3, e_4) = \begin{cases} h(p; e_1-e_2, e_3, e_4) - (p+1-e_1-e_2) : & e_1 \neq e_2, \\ h(p; e_1-e_2, e_3, e_4) - (p+1-e_1-e_2)/2 : & e_1 = e_2. \end{cases}$$

Corollary 6.5 gives a more general result including the case that some of the  $e_i$  are even, but in some cases we also compute the  $p$ -Hurwitz number only up to a factor 2. Note that there is an explicit formula for  $h(p; e_1, e_2, e_3, e_4)$  and  $h(p; e_1-e_2, e_3, e_4)$ ; see Theorem 2.1 and Lemma 2.2 below.

Our technique involves the use of “admissible covers,” which are certain covers between degenerate curves (see Section 2). Admissible covers provide a compactification of the space of covers of smooth curves in characteristic 0, but in positive characteristic this is not the case, and it is an interesting question when, under a given degeneration of the base, a cover of smooth curves does in fact have an admissible cover as a limit. In this case we say the smooth cover has *good degeneration*. In [2] one finds examples of covers with generic branch points without good degeneration.

In contrast, our technique for proving Theorem 1.1 simultaneously shows that many of the examples we consider have good degeneration.

**Theorem 1.3.** *Given odd integers  $1 < e_1 \leq e_2 \leq e_3 \leq e_4 < p$  with  $\sum_i e_i = 2p + 2$ , every cover of type  $(p; e_1, e_2, e_3, e_4)$  with generic branch points  $(0, 1, \lambda, \infty)$  has good degeneration under the degeneration sending  $\lambda$  to  $\infty$ .*

As with Theorem 1.2, our methods do not give a complete answer in some cases with even  $e_i$ , but we do prove a more general result in Theorem 8.2.

Building on the work of Raynaud [16], Wewers uses the theory of stable reduction in [20] to give formulas for the number of covers with three branch points and having Galois closure of degree strictly divisible by  $p$  which have bad reduction to characteristic  $p$ . In [5], some  $p$ -Hurwitz numbers are calculated using the existence portion of Wewers’ theorems, but these are in cases which are rigid (meaning the classical Hurwitz number is 1) or very close to rigid, so one does not have to carry out calculations with Wewers’ formulas. In [17], Selander uses the full statement of Wewers’ formulas to compute some examples in small degree. Our result in Theorem 1.2 is the first explicit calculation of an infinite family of  $p$ -Hurwitz numbers which fully uses Wewers’ formulas, and its proof occupies the bulk of the present paper.

We begin in Sections 2 and 3 by reviewing the situation in characteristic 0 and some group-theoretic background. We then recall the theory of stable reduction in Section 4. In order to apply Wewers’ formulas, in Section 5 we analyze the possible structures of the stable reductions which arise, and then in Section 6 we apply Wewers’ formulas to compute the number of smooth covers with a given stable reduction. Here we are forced to use a trick comparing the number of covers in the case of interest to the number in a related case where we know all covers have bad reduction. In Section 7 we then apply Corollary 6.5 as well as the formulas for  $h_p(d; e_1, e_2, e_3)$  of [14] and the classical Hurwitz number calculations in [12] to

estimate the number of admissible covers in characteristic  $p$ . This provides a sufficient lower bound on  $h_p(p; e_1, e_2, e_3, e_4)$ . Finally, we use the techniques of [3], again based on stable reduction, to directly prove in Section 8 that  $h_p(p; e_1, e_2, e_3, e_4)$  is bounded above by  $h(p; e_1, e_2, e_3, e_4) - p$ . We thus conclude Theorems 1.1 and 1.3.

We would like to thank Peter Müller, Björn Selander and Robert Guralnick for helpful discussions.

## 2. THE CHARACTERISTIC-0 SITUATION

In this paper, we consider covers  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  branched at  $r$  ordered points  $Q_1, \dots, Q_r$  of fixed *ramification type*  $(d; C_1, \dots, C_r)$ , where  $d$  is the degree of  $f$  and  $C_i = e_1(i) \cdots e_{s_i}(i)$  is a conjugacy class in  $S_d$ . This means that there are  $s_i$  ramification points in the fiber  $f^{-1}(Q_i)$ , with ramification indices  $e_j(i)$ . The *Hurwitz number*  $h(d; C_1, \dots, C_r)$  is the number of covers of fixed ramification type over  $\mathbb{C}$ , up to isomorphism. This number does not depend on the position of the branch points.

Riemann's Existence Theorem implies that the Hurwitz number  $h(d; C_1, \dots, C_r)$  is the cardinality of the set of *Hurwitz factorizations* defined as

$$\{(g_1, \dots, g_r) \in C_1 \times \dots \times C_r \mid \langle g_i \rangle \subset S_d \text{ transitive, } \prod_i g_i = 1\} / \sim,$$

where  $\sim$  denotes uniform conjugacy by  $S_d$ .

The group  $\langle g_i \rangle$  is called the *monodromy group* of the corresponding cover. For a fixed monodromy group  $G$ , a variant equivalence relation is given by  *$G$ -Galois covers*, where we work with Galois covers together with a fixed isomorphism of the Galois group to  $G$ . The group-theoretic interpretation is then that the  $g_i$  are in  $G$  (with the action on a fiber recovered by considering  $G$  as a subgroup of  $S_{|G|}$ ), and the equivalence relation  $\sim_G$  is uniform conjugacy by  $G$ . To contrast with the  $G$ -Galois case, we sometimes emphasize that we are working up to  $S_d$ -conjugacy by referring to the corresponding covers as *mere covers*.

In this paper, we are mainly interested in the *pure-cycle* case, where every  $C_i$  is the conjugacy class in  $S_d$  of a single cycle. In this case, we write  $C_i = e_i$ , where  $e_i$  is the length of the cycle. A cover  $f : Y \rightarrow \mathbb{P}^1$  over  $\mathbb{C}$  of ramification type  $(d; e_1, e_2, \dots, e_r)$  has genus  $g(Y) = 0$  if and only if  $\sum_{i=1}^r e_i = 2d - 2 + r$ .

Giving closed formulae for Hurwitz numbers may get very complicated, even in characteristic zero. The following result from [12] illustrates that the genus-0 pure-cycle case is more tractable than the general case, as one may give closed formulae for the Hurwitz numbers, at least if the number  $r$  of branch points is at most 4.

**Theorem 2.1.** *Under the hypothesis  $\sum_{i=1}^r e_i = 2d - 2 + r$ , we have the following.*

- (a)  $h(d; e_1, e_2, e_3) = 1$ .
- (b)  $h(d; e_1, e_2, e_3, e_4) = \min_i (e_i(d + 1 - e_i))$ .
- (c) *Let  $f : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be a cover of ramification type  $(d; e_1, e_2, \dots, e_r)$  with  $r \geq 3$ . The Galois group of the Galois closure of  $f$  is either  $S_d$  or  $A_d$  unless  $(d; e_1, e_2, \dots, e_r) = (6; 4, 4, 5)$  in which case the Galois group is  $S_5$  acting transitively on 6 letters.*

These statements are Lemma 2.1, Theorem 4.2, and Theorem 5.3 of [12]. We mention that Boccara ([1]) proves a partial generalization of Theorem 2.1.(a). He gives a necessary and sufficient condition for  $h(d; C_1, C_2, \ell)$  to be nonzero in the

case that  $C_1, C_2$  are arbitrary conjugacy classes of  $S_d$  and only  $C_3 = \ell$  is assumed to be the conjugacy class of a single cycle.

Later in our analysis we will be required to study covers of type  $(d; e_1-e_2, e_3, e_4)$ , so we mention a result which is not stated explicitly in [12], but which follows easily from the arguments therein. We will only use the case that  $e_4 = d$ , but we state the result in general since the argument is the same.

**Lemma 2.2.** *Given  $e_1, e_2, e_3, e_4$  and  $d$  with  $2d + 2 = \sum_i e_i$  and  $e_1 + e_2 \leq d$ , if  $e_1 \neq e_2$  we have*

$$h(d; e_1-e_2, e_3, e_4) = (d + 1 - e_1 - e_2) \min(e_1, e_2, d + 1 - e_3, d + 1 - e_4),$$

and if  $e_1 = e_2$  we have

$$h(d; e_1-e_2, e_3, e_4) = \lceil \frac{1}{2} (d + 1 - e_1 - e_2) \min(d + 1 - e_3, d + 1 - e_4) \rceil.$$

Note that this number is always positive. In particular, when  $e_4 = d$  we have

$$h(d; e_1-e_2, e_3, d) = \begin{cases} d + 1 - e_1 - e_2 & \text{if } e_1 \neq e_2, \\ (d + 2 - e_1 - e_2)/2 & \text{if } e_1 = e_2, d \text{ even}, \\ (d + 1 - e_1 - e_2)/2 & \text{if } e_1 = e_2, d \text{ odd}. \end{cases}$$

*Proof.* Without loss of generality, we may assume that  $e_1 \leq e_2$  and  $e_3 \leq e_4$ . Thus, we want to prove that  $h(d; e_1-e_2, e_3, e_4)$  is given by the smaller of  $e_1(d + 1 - e_1 - e_2)$  and  $(d + 1 - e_4)(d + 1 - e_1 - e_2)$  when  $e_1 \neq e_2$ , by  $((d + 1 - e_4)(d + 1 - e_1 - e_2) + 1)/2$  when  $e_1 = e_2$  and all of  $d, e_3, e_4$  are even, and by  $(d + 1 - e_4)(d + 1 - e_1 - e_2)/2$  otherwise. Even though we do not assume  $e_2 \leq e_3$ , this formula still follows from the argument of Theorem 4.2.(ii) of [12]. The first observations to make are that since  $e_1 + e_2 \leq d$ , we have  $e_3 + e_4 \geq d + 2$ , and it follows that although we may not have  $e_2 \leq e_3$ , we have  $e_1 < e_4$ . Moreover, we have  $e_1 + e_3 \leq d + 1$  and  $e_2 + e_4 \geq d + 1$ . We are then able to check that the Hurwitz factorizations  $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$  described in case (ii) of *loc. cit.* still give valid Hurwitz factorizations  $(g_1, g_2, g_3)$  by setting  $g_1 = \sigma_1 \sigma_2$ , just as in Proposition 4.7 of *loc. cit.* Moreover, just as in Proposition 4.9 of *loc. cit.* we find that every Hurwitz factorization must be one of the enumerated ones.

It remains to consider when two of the described possibilities yield equivalent Hurwitz factorizations. If  $e_1 \neq e_2$ , we can extract  $\sigma_1$  and  $\sigma_2$  as the disjoint cycles (of distinct orders) in  $g_1$ , so we easily see that the proof of Proposition 4.8 of *loc. cit.* is still valid. Thus the Hurwitz number is simply the number of possibilities enumerated in Theorem 4.2 (ii) of [12], which is the minimum of  $e_1(d + 1 - e_1 - e_2)$  and  $(d + 1 - e_4)(d + 1 - e_1 - e_2)$ , as desired.

Now suppose  $e_1 = e_2$ . We then check easily that  $e_1 + e_4 \geq d + 1$ , so that the number of enumerated possibilities is  $(d + 1 - e_4)(d + 1 - e_1 - e_2)$ . Here, we see that we potentially have a given Hurwitz factorization  $(g_1, g_2, g_3)$  being simultaneously equivalent to two of the enumerated possibilities, since  $\sigma_1$  and  $\sigma_2$  can be switched. Indeed, the argument of Proposition 4.8 of *loc. cit.* describing how to intrinsically recover the parameters  $m, k$  of Theorem 4.2 (ii) of *loc. cit.* lets us compute how  $m, k$  change under switching  $\sigma_1$  and  $\sigma_2$ , and we find that the pair  $(m, k)$  is sent to  $(e_3 + 2e_4 - d - m, e_3 + e_4 - d - k)$ . We thus find that each Hurwitz factorization is equivalent to two distinct enumerated possibilities, with the exception that if  $d$  and  $e_4$  (and therefore necessarily  $e_3$ ) are even, the Hurwitz factorization corresponding

to  $(m, k) = ((e_3 + 2e_4 - d)/2, (e_3 + e_4 - d)/2)$  is not equivalent to any other. We therefore conclude the desired statement.  $\square$

We now explain how Theorem 4.2 of [12] can be understood in terms of degenerations. Harris and Mumford [10] developed the theory of *admissible covers*, giving a description of the behavior of branched covers under degeneration. Admissible covers in the case we are interested in may be described geometrically as follows: let  $X_0$  be the reducible curve consisting of two smooth rational components  $X_0^1$  and  $X_0^2$  joined at a single node  $Q$ . We suppose we have points  $Q_1, Q_2$  on  $X_0^1$ , and  $Q_3, Q_4$  on  $X_0^2$ . An *admissible cover* of type  $(d; C_1, C_2, *, C_3, C_4)$  is then a connected, finite flat cover  $f_0 : Y_0 \rightarrow X_0$  which is étale away from the preimage of  $Q$  and the  $Q_i$ , and if we denote by  $Y_0^1 \rightarrow X_0^1$  and  $Y_0^2 \rightarrow X_0^2$  the (possibly disconnected) covers of  $X_0^1$  and  $X_0^2$  induced by  $f_0$ , we require also that  $Y_0^1 \rightarrow X_0^1$  has ramification type  $(d; C_1, C_2, C)$  for  $Q_1, Q_2, Q$  and  $Y_0^2 \rightarrow X_0^2$  has ramification type  $(d; C, C_3, C_4)$  for  $Q, Q_3, Q_4$ , for some conjugacy class  $C$  in  $S_d$ , and furthermore that for  $P \in f_0^{-1}(Q)$ , the ramification index of  $f_0$  at  $P$  is the same on  $Y_0^1$  and  $Y_0^2$ . In characteristic  $p$ , we further have to require that ramification above the node is tame. We refer to  $Y_0^1 \rightarrow X_0^1$  and  $Y_0^2 \rightarrow X_0^2$  as the *component covers* determining  $f_0$ . When we wish to specify the class  $C$ , we say the admissible cover is of type  $(d, C_1, C_2, *_C, C_3, C_4)$ .

The two basic theorems on admissible covers concern degeneration and smoothing. First, in characteristic 0, or when the monodromy group has order prime to  $p$ , if a family of smooth covers of type  $(d; C_1, C_2, C_3, C_4)$  with branch points  $(Q_1, Q_2, Q_3, Q_4)$  is degenerated by sending  $Q_3$  to  $Q_4$ , the limit is an admissible cover of type  $(d; C_1, C_2, *, C_3, C_4)$ . On the other hand, given an admissible cover of type  $(d; C_1, C_2, *, C_3, C_4)$ , irrespective of characteristic there is a deformation to a cover of smooth curves, which then has type  $(d; C_1, C_2, C_3, C_4)$ . Such a deformation is not unique in general; we call the number of smooth covers arising as smoothings of a given admissible cover (for a fixed smoothing of the base) the *multiplicity* of the admissible cover.

Suppose we have a family of covers  $f : X \rightarrow Y$ , with smooth generic fiber  $f_1 : X_1 \rightarrow Y_1$ , and admissible special fiber  $f_0 : X_0 \rightarrow Y_0$ . If we choose local monodromy generators for  $\pi_1^{\text{tame}}(Y_1 \setminus \{Q_1, Q_2, Q_3, Q_4\})$  which are compatible with the degeneration to  $Y_0$ , we then find that if we have a branched cover of  $Y_1$  corresponding to a Hurwitz factorization  $(g_1, g_2, g_3, g_4)$ , the induced admissible cover of  $Y_0$  will have monodromy given by  $(g_1, g_2, \rho)$  over  $Y_0^1$  and  $(\rho^{-1}, g_3, g_4)$  over  $Y_0^2$ , where  $\rho = g_3 g_4$ . The multiplicity of the admissible cover arises because it may be possible to apply different simultaneous conjugations to  $(g_1, g_2, \rho)$  and to  $(\rho^{-1}, g_3, g_4)$  while maintaining the relationship between  $\rho$  and  $\rho^{-1}$ . It is well-known that when  $\rho$  is a pure-cycle of order  $m$ , the admissible cover has multiplicity  $m$ , although we recover this fact independently in our situation as part of the Hurwitz number calculation of [12].

To calculate more generally the multiplicity of an admissible cover of the above type, we define the action of the braid operator  $Q_3$  on the set of Hurwitz factorizations as

$$Q_3 \cdot (g_1, g_2, g_3, g_4) = (g_1 g_2 g_1 g_2^{-1} g_1^{-1}, g_1 g_2 g_1^{-1}, g_3, g_4).$$

One easily checks that  $Q_3 \cdot \bar{g}$  is again a Hurwitz factorization of the same ramification type as  $\bar{g}$ . The multiplicity of a given admissible cover is the length of the orbit of  $Q_3$  acting on the corresponding Hurwitz factorization.

In this context, we can give the following sharper statement of Theorem 2.1 (b), phrased in somewhat different language in [12].

**Theorem 2.3.** *Given a genus-0 ramification type  $(d; e_1, e_2, e_3, e_4)$ , with  $e_1 \leq e_2 \leq e_3 \leq e_4$  the only possibilities for an admissible cover of type  $(d; e_1, e_2, *, e_3, e_4)$  are type  $(d; e_1, e_2, *_m, e_3, e_4)$  or type  $(d; e_1, e_2, *_{e_1-e_2}, e_3, e_4)$ .*

(a) *Fix  $m \geq 1$ . There is at most one admissible cover of type  $(d; e_1, e_2, *_m, e_3, e_4)$ , and if such a cover exists, it has multiplicity  $m$ .*

(i) *Suppose that  $d+1 \leq e_2 + e_3$ . There exists an admissible cover of type  $(d; e_1, e_2, *_m, e_3, e_4)$  if and only if*

$$e_2 - e_1 + 1 \leq m \leq 2d + 1 - e_3 - e_4, \quad m \equiv e_2 - e_1 + 1 \pmod{2}.$$

(ii) *Suppose that  $d+1 \geq e_2 + e_3$ . There exists an admissible cover of type  $(d; e_1, e_2, *_m, e_3, e_4)$  if and only if*

$$e_4 - e_3 + 1 \leq m \leq 2d + 1 - e_3 - e_4, \quad m \equiv e_2 - e_1 + 1 \pmod{2}.$$

(b) *Admissible covers of type  $(d; e_1, e_2, *_{e_1-e_2}, e_3, e_4)$  have multiplicity 1. The component cover of type  $(d; e_1, e_2, e_1-e_2)$  is uniquely determined, so the admissible cover is determined by its second component cover and the gluing over the node. Moreover, the gluing over the node is unique when  $e_1 \neq e_2$ . When  $e_1 = e_2$ , there are always two possibilities for gluing except for a single admissible cover in the case that  $e_3, e_4$ , and  $d$  are all even.*

*The number of admissible covers of this type is*

$$\begin{cases} e_1(d+1-e_1-e_2) & \text{if } d+1 \leq e_2 + e_3, \\ (e_3 + e_4 - d - 1)(d+1-e_4) & \text{if } d+1 \geq e_2 + e_3. \end{cases}$$

*Proof.* We briefly explain how this follows from Theorem 4.2 of [12]. As stated above, the possibilities for admissible covers are determined by pairs  $(g_1, g_2, \rho)$ ,  $(\rho^{-1}, g_3, g_4)$  where  $(g_1, g_2, g_3, g_4)$  is a Hurwitz factorization of type  $(d; e_1, e_2, e_3, e_4)$ . *Loc. cit.* immediately implies that  $\rho$  is always either a single cycle of length  $m \geq 1$  or the product of two disjoint cycles.

For (a), we find from part (i) of *loc. cit.* that the ranges for  $m$  (which is  $e_3 + e_4 - 2k$  is the notation of *loc. cit.*) are as asserted, and that for a given  $m$ , the number of possibilities with  $\rho$  an  $m$ -cycle is precisely  $m$ , when counted with multiplicity. On the other hand, in this case both component covers are three-point pure-cycle covers, and thus uniquely determined (see Theorem 2.1 (a)). Thus the admissible cover is unique in this case, with multiplicity  $m$ .

For (b), we see by inspection of the description of part (ii) of *loc. cit.* that  $g_1$  is disjoint from  $g_2$ . It immediately follows that the braid action is trivial, so the multiplicity is always 1, and the asserted count of covers follows immediately from the proof of Proposition 4.10 of *loc. cit.* Moreover, the component cover of type  $(d; e_1, e_2, e_1-e_2)$  is a disjoint union of covers of type  $(e_1; e_1, e_1)$  and  $(e_2; e_2, e_2)$  (as well as  $d - e_1 - e_2$  copies of the trivial cover), so it is uniquely determined, as asserted. Furthermore, we see that the second component cover is always a single connected cover of degree  $d$ , and  $g_1, g_2$  are recovered as the disjoint cycles of  $\rho^{-1}$ , so the gluing is unique when  $e_1 \neq e_2$ . When  $e_1 = e_2$ , it is possible to swap  $g_1$  and  $g_2$ , so we see that there are two possibilities for gluing. The argument of Lemma 2.2 shows that we do in fact obtain two distinct admissible covers in this way, except for a single cover occurring when  $e_3, e_4$  and  $d$  are all even.  $\square$



## 3. GROUP THEORY

In several contexts, we will have to calculate monodromy groups other than those treated by Theorem 2.1 (c). We will also have to pass between counting mere covers and counting  $G$ -Galois covers. In this section, we give basic group-theoretic results to address these topics.

Since we restrict our attention to covers of prime degree, the following proposition will be helpful.

**Proposition 3.1.** *Let  $p$  be a prime number and  $G$  a transitive group on  $p$  letters. Suppose that  $G$  contains a pure cycle of length  $1 < e < p - 2$ . Then  $G$  is either  $A_p$  or  $S_p$ .*

*Moreover, if  $e = p - 2$ , and  $G$  is neither  $A_p$  nor  $S_p$ , then  $p = 2^r + 1$  for some  $r$ , and  $G$  contains a unique minimal normal subgroup isomorphic to  $\mathrm{PSL}_2(2^r)$ , and is contained in  $\mathrm{PTL}_2(2^r) \simeq \mathrm{PSL}_2(2^r) \rtimes \mathbb{Z}/r\mathbb{Z}$ . If  $e = p - 1$ , and  $G$  is not  $S_p$ , then  $G = \mathbb{F}_p \rtimes \mathbb{F}_p^*$ .*

Note that this does not contradict the exceptional case  $d = 6$  and  $G = S_5$  in Theorem 2.1 (c), since we assume that the degree  $d$  is prime.

*Proof.* Since  $p$  is prime,  $G$  is necessarily primitive, and a theorem usually attributed to Marggraff ([11]) then states that  $G$  is at least  $(p - e + 1)$ -transitive. When  $e \leq p - 2$ , we have that  $p - e + 1 \geq 3$ . The 2-transitive permutation groups have been classified by Cameron (Section 5 of [7]). Specifically,  $G$  has a unique minimal normal subgroup which is either elementary abelian or one of several possible simple groups. Since  $G$  is at least 3-transitive, one easily checks that the elementary abelian case is not possible: indeed, one checks directly that if a subgroup of a 3-transitive group inside  $S_p$  contains an element of prime order  $\ell$ , then it is not possible for all its conjugates to commute with one another. Similarly, most possibilities in the simple case cannot be 3-transitive. If  $G$  is not  $S_p$  or  $A_p$ , then  $G$  must have a unique minimal normal subgroup  $N$  which is isomorphic to a Mathieu group  $M_{11}, M_{23}$ , or to  $N = \mathrm{PSL}_2(2^r)$ . We then have that  $G$  is a subgroup of  $\mathrm{Aut}(N)$ . For  $N = M_{11}, M_{23}$ , we have  $N = G = \mathrm{Aut}(N)$ , and it is easy to check that the Mathieu groups  $M_{11}$  and  $M_{23}$  do not contain any single cycles of order less than  $p$ , for example with the computer algebra package GAP. Therefore these cases do not occur. The group  $\mathrm{PSL}_2(2^r)$  can only occur if  $p = 2^r + 1$ . In this case, we have that  $G$  is a subgroup of  $\mathrm{Aut}(\mathrm{PSL}_2(2^r)) = \mathrm{PTL}_2(2^r)$  and  $G$  is at most 3-transitive, so we have  $e = p - 2$ , as desired.

Finally, if  $e = p - 1$ , Müller has classified transitive permutation groups containing  $(p - 1)$ -cycles in Theorem 6.2 of [13], and we see that the only possibility in prime degree other than  $S_p$  is  $\mathbb{F}_p \rtimes \mathbb{F}_p^*$ , as asserted.  $\square$

We illustrate the utility of the proposition with:

**Corollary 3.2.** *Fix  $e_1, e_2, e_3, e_4$  with  $2 \leq e_i \leq p$  for each  $i$ , and  $e_1 + e_2 \leq p$ . For  $p > 5$ , any genus-0 cover of type  $(p; e_1 - e_2, e_3, e_4)$  has monodromy group  $S_p$  or  $A_p$ , with the latter case occurring precisely when  $e_3$  and  $e_4$  are odd, and  $e_1 + e_2$  is even. For  $p = 5$ , the only exceptional case is type  $(5; 2-2, 4, 4)$ , where the monodromy group is  $\mathbb{F}_5 \rtimes \mathbb{F}_5^*$ .*

*Proof.* Without loss of generality, we assume  $e_1 \leq e_2$  and  $e_3 \leq e_4$ . Applying Proposition 3.1, it is clear that the only possible exception occurs for types with

$e_3, e_4 \geq p-2$ . We thus have to treat types  $(p; 3-3, p-2, p-2)$ ,  $(p; 2-4, p-2, p-2)$ ,  $(p; 2-2, p-2, p)$ ,  $(p; 2-3, p-2, p-1)$ , and  $(p; 2-2, p-1, p-1)$ . The fourth case cannot be exceptional since  $G$  contains both a  $(p-2)$ -cycle and a  $(p-1)$ -cycle, and the last case also is ruled out for  $p > 5$  because  $\mathbb{F}_p \rtimes \mathbb{F}_p^*$  does not contain a 2-2-cycle.

For the first three cases, we must have that  $p = 2^r + 1$  for some  $r$  and  $G$  is a subgroup of  $\Gamma := \text{P}\Gamma\text{L}_2(2^r)$ . Since  $p = 2^r + 1$  is a Fermat prime number, we have that  $r$  is a power of 2. Moreover, since  $\text{PSL}_2(4) = A_5$  as permutation groups in  $S_5$ , we may assume  $r \geq 4$ . Since  $r$  is even, any element of order 3 in  $\Gamma \cong \text{PSL}_2(2^r) \rtimes \mathbb{Z}/r\mathbb{Z}$  must lie inside  $\text{PSL}_2(2^r)$ , and a non-trivial element of this group can fix at most 2 letters. Thus, in order to contain a 3-3-cycle, we would have to have  $6 \leq p = 2^r + 1 \leq 8$ , which contradicts the hypothesis  $r \geq 4$ . This rules out the first case. In the second case, if we square the 2-4-cycle we obtain a 2-2-cycle. To complete the argument for both the second and third cases it is thus enough to check directly that if  $r > 4$ , an element of order 2 cannot fix precisely  $p-4$  letters, ruling out a 2-2-cycle in this case. It remains only to check directly that  $\text{P}\Gamma\text{L}_2(16)$  does not contain a 2-2-cycle, which one can do directly with GAP.  $\square$

Because the theory of stable reduction is developed in the  $G$ -Galois context, it is convenient to be able to pass back and forth between the context of mere covers and of  $G$ -Galois covers. The following easy result relates the number of mere covers to the number of  $G$ -Galois covers in the case we are interested in.

**Lemma 3.3.** *Let  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a (mere) cover of degree  $d$  with monodromy group  $G = A_d$  (respectively,  $S_d$ ). Then the number of  $G$ -Galois structures on the Galois closure of  $f$  is exactly 2 (respectively, 1).*

*Proof.* The case that  $G = S_d$  is clear, since conjugacy by  $S_d$  is then the same as conjugacy by  $G$ . Suppose  $G = A_d$ , and let  $X = \{(g_1, \dots, g_r) \mid \prod_i g_i = 1, \langle g_i \rangle = d\}$ . Since the centralizer  $C_{S_d}(A_d)$  of  $A_d$  in  $S_d$  is trivial, it follows that  $S_d$  acts freely on  $X$ , so the number of elements in  $X_f \subseteq X$  corresponding to  $f$  as a mere cover is  $|S_d|$ . Since the center of  $G = A_d$  is trivial,  $G$  also acts freely on  $X$ , and  $X_f$  breaks into two equivalence classes of  $G$ -Galois covers, each of size  $|A_d|$ .  $\square$

#### 4. STABLE REDUCTION

In this section, we recall some generalities on stable reduction of Galois covers of curves, and prove a few simple lemmas as a prelude to our main calculations. The main references for this section are [20] and [3]. Since these sources only consider the case of  $G$ -Galois covers, we restrict to this situation here as well. Lemma 3.3 implies that we may translate results on good or bad reduction of  $G$ -Galois covers to results on the stable reduction of the mere covers, so this is no restriction.

Let  $R$  be a discrete valuation ring with fraction field  $K$  of characteristic zero and residue field an algebraically closed field  $k$  of characteristic  $p > 0$ . Let  $f : V = \mathbb{P}_K^1 \rightarrow X = \mathbb{P}_K^1$  be a degree- $p$  cover branched at  $r$  points  $Q_1 = 0, Q_2 = 1, \dots, Q_r = \infty$  over  $K$  with ramification type  $(p; C_1, \dots, C_r)$ . For the moment, we do not assume that the  $C_i$  are the conjugacy classes of a single cycle. We denote the Galois closure of  $f$  by  $g : Y \rightarrow \mathbb{P}^1$  and let  $G$  be its Galois group. Note that  $G$  is a transitive subgroup of  $S_p$ , and thus has order divisible by  $p$ . Write  $H := \text{Gal}(Y, V)$ , a subgroup of index  $p$ .

We suppose that  $Q_i \not\equiv Q_j \pmod{p}$ , for  $i \neq j$ , in other words, that  $(X; \{Q_i\})$  has good reduction as a marked curve. We assume moreover that  $g$  has bad reduction to



characteristic  $p$ , and denote by  $\bar{g} : \bar{Y} \rightarrow \bar{X}$  its *stable reduction*. The stable reduction  $\bar{g}$  is defined as follows. After replacing  $K$  by a finite extension, there exists a unique stable model  $\mathcal{Y}$  of  $Y$  as defined in [20]. We define  $\mathcal{X} = \mathcal{Y}/G$ . The stable reduction  $\bar{g} : \bar{Y} := \mathcal{Y} \otimes_R k \rightarrow \bar{X} := \mathcal{X} \otimes_R k$  is a finite map between semistable curves in characteristic  $p$ ; we call such maps *stable  $G$ -maps*. We refer to [20], Definition 2.14, for a precise definition.

Roughly speaking, the theory of stable reduction proceeds in two steps: first, one understands the possibilities for stable  $G$ -maps, and then one counts the number of characteristic-0 covers reducing to each stable  $G$ -map.

We begin by describing what the stable reduction must look like. Since  $(X; Q_i)$  has good reduction to characteristic  $p$ , there exists a model  $\mathcal{X}_0 \rightarrow \text{Spec}(R)$  such that the  $Q_i$  extend to disjoint sections. There is a unique irreducible component  $\bar{X}_0$  of  $\bar{X}$ , called the *original component*, on which the natural contraction map  $\bar{X} \rightarrow \mathcal{X}_0 \otimes_R k$  is an isomorphism. The restriction of  $\bar{g}$  to  $\bar{X}_0$  is inseparable.

Let  $\mathbb{B} \subseteq \{1, 2, \dots, r\}$  consist of those indices  $i$  such that  $C_i$  is not the conjugacy class of a  $p$ -cycle. For  $i \in \mathbb{B}$ , we have that  $Q_i$  specializes to an irreducible component  $\bar{X}_i \neq \bar{X}_0$  of  $\bar{X}$ . The restriction of  $\bar{g}$  to  $\bar{X}_i$  is separable, and  $\bar{X}_i$  intersects the rest of  $\bar{X}$  in a single point  $\xi_i$ . Let  $\bar{Y}_i$  be an irreducible component of  $\bar{Y}$  above  $\bar{X}_i$ , and write  $\bar{g}_i : \bar{Y}_i \rightarrow \bar{X}_i$  for the restriction of  $\bar{g}$  to  $\bar{Y}_i$ . We denote by  $G_i$  the decomposition group of  $\bar{Y}_i$ . We call the components  $\bar{X}_i$  (resp. the covers  $\bar{g}_i$ ) for  $i \in \mathbb{B}$  the *primitive tails* (resp. the *primitive tail covers*) associated with the stable reduction. The following definition gives a characterization of those covers that can arise as primitive tail covers (compare to [20], Section 2.2).

**Definition 4.1.** Let  $k$  be an algebraically closed field of characteristic  $p > 0$ . Let  $C$  be a conjugacy class of  $S_p$  which is not the class of a  $p$ -cycle. A *primitive tail cover* of ramification type  $C$  is a  $G$ -Galois cover  $\varphi_C : T_C \rightarrow \mathbb{P}_k^1$  defined over  $k$  which is branched at exactly two points  $0, \infty$ , satisfying the following conditions.

- (a) The Galois group  $G_C$  of  $\varphi_C$  is a subgroup of  $S_p$  and contains a subgroup  $H$  of index  $p$  such that  $\bar{T}_C := T_C/H$  has genus 0.
- (b) The induced map  $\bar{\varphi}_C : \bar{T}_C \rightarrow \mathbb{P}^1$  is tamely branched at  $x = 0$ , with conjugacy class  $C$ , and wildly branched at  $x = \infty$ .

If  $\varphi$  is a tail cover, we let  $h = h(\varphi)$  be the conductor and  $pm = pm(\varphi)$  the ramification index of a wild ramification point.

We say that two primitive tail covers  $\varphi_i : T_i \rightarrow \mathbb{P}_k^1$  are *isomorphic* if there exists a  $G$ -equivariant isomorphism  $\iota : T_1 \rightarrow T_2$ . Note that we do not require an isomorphism to send  $\bar{T}_1$  to  $\bar{T}_2$ .

Note that an isomorphism  $\iota$  of primitive tail covers may be completed into a commuting square

$$\begin{array}{ccc} T_1 & \xrightarrow{\iota} & T_2 \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

Note also that the number of primitive tail covers of fixed ramification type is finite.

Since  $p$  strictly divides the order of the Galois group  $G_C$ , we conclude that  $m$  is prime to  $p$ . The invariants  $h, m$  describe the wild ramification of the tail cover  $\varphi_C$ . The integers  $h$  and  $m$  only depend on the conjugacy class  $C$ . In Section 5, we will show this if  $C$  is the class of a single cycle or the product of 2 disjoint cycles,

but this holds more generally. In the more general set-up of [20], Definition 2.9 it is required that  $\sigma := h/m < 1$  as part of the definition of primitive tail cover. We will see that in our situation this follows from (a). Moreover, we will show that  $\gcd(h, m) = 1$  (Lemma 5.1). Summarizing, we find that  $(h, m)$  satisfy:

$$(4.1) \quad m \mid (p-1), \quad 1 \leq h < m, \quad \gcd(h, m) = 1.$$

In the more general set-up of [20] there also exists so-called new tails, which satisfy  $\sigma > 1$ . The following lemma implies that these do not occur in our situation.

**Lemma 4.2.** *The curve  $\bar{X}$  consists of at most  $r+1$  irreducible components: the original component  $\bar{X}_0$  and primitive tails  $\bar{X}_i$  for all  $i \in \mathbb{B}$ .*

*Proof.* In the case that  $r = 3$  this is proved in [20], Section 4.4, using that the cover is the Galois closure of a genus-0 cover of degree  $p$ . The general case is a straightforward generalization.  $\square$

It remains to discuss the restriction of  $\bar{g}$  to the original component  $\bar{X}_0$ . As mentioned above, this restriction is inseparable, and it is described by a so-called deformation datum ([20], Section 1.3).

In order to describe deformation data, we set some notation. Let  $\bar{Q}_i$  be the limit on  $\bar{X}_0$  of the  $Q_i$  for  $i \notin \mathbb{B}$ , and set  $\bar{Q}_i = \xi_i$  for  $i \in \mathbb{B}$ .

**Definition 4.3.** Let  $k$  be an algebraically closed field of characteristic  $p$ . A *deformation datum* is a pair  $(Z, \omega)$ , where  $Z$  is a smooth projective curve over  $k$  together with a finite Galois cover  $g : Z \rightarrow X = \mathbb{P}_k^1$ , and  $\omega$  is a meromorphic differential form on  $Z$  such that the following conditions hold.

- (a) Let  $H$  be the Galois group of  $Z \rightarrow X$ . Then

$$\beta^* \omega = \chi(\beta) \cdot \omega, \quad \text{for all } \beta \in H.$$

Here  $\chi : H \rightarrow \mathbb{F}_p^\times$  is an injective character.

- (b) The differential form  $\omega$  is either logarithmic, i.e. of the form  $\omega = df/f$ , or exact, i.e. of the form  $df$ , for some meromorphic function  $f$  on  $Z$ .

Note that the cover  $Z \rightarrow X$  is necessarily cyclic.

To a  $G$ -Galois cover  $g : Y \rightarrow \mathbb{P}^1$  with bad reduction, we may associate a deformation datum, as follows. Choose an irreducible component  $\bar{Y}_0$  of  $\bar{Y}$  above the original component  $\bar{X}_0$ . Since the restriction  $\bar{g}_0 : \bar{Y}_0 \rightarrow \bar{X}_0$  is inseparable and  $G \subset S_p$ , it follows that the inertia group  $I$  of  $\bar{Y}_0$  is cyclic of order  $p$ , i.e. a Sylow  $p$ -subgroup of  $G$ . Since the inertia group is normal in the decomposition group, the decomposition group  $G_0$  of  $\bar{Y}_0$  is a subgroup of  $N_{S_p}(I) \simeq \mathbb{Z}/p\mathbb{Z} \rtimes_\chi \mathbb{Z}/p\mathbb{Z}^*$ , where  $\chi : \mathbb{Z}/p\mathbb{Z}^* \rightarrow \mathbb{Z}/p\mathbb{Z}^*$  is an injective character. It follows that the map  $\bar{g}_0$  factors as  $\bar{g}_0 : \bar{Y}_0 \rightarrow \bar{Z}_0 \rightarrow \bar{X}_0$ , where  $\bar{Y}_0 \rightarrow \bar{Z}_0$  is inseparable of degree  $p$  and  $\bar{Z}_0 \rightarrow \bar{X}_0$  is separable. We conclude that the Galois group  $H_0 := \text{Gal}(\bar{Z}_0, \bar{X}_0)$  is a subgroup of  $\mathbb{Z}/p\mathbb{Z}^* \simeq \mathbb{Z}/(p-1)\mathbb{Z}$ . In particular, it follows that

$$(4.2) \quad G_0 \simeq I \rtimes_\chi H_0.$$

The inseparable map  $\bar{Y}_0 \rightarrow \bar{Z}_0$  is characterized by a differential form  $\omega$  on  $\bar{Z}_0$  satisfying the properties of Definition 4.3, see [20], Section 1.3.2.

The differential form  $\omega$  is logarithmic if  $\bar{Y}_0 \rightarrow \bar{Z}_0$  is a  $\mu_p$ -torsor and exact if this map is an  $\alpha_p$ -torsor. A differential form is logarithmic if and only if it is fixed by the Cartier operator  $\mathcal{C}$  and exact if and only if it is killed by  $\mathcal{C}$ . (See for example [8], exercise 9.6, for the definition of the Cartier operator and an outline of these

properties.) Wewers ([19], Lemma 2.8) shows that in the case of covers branched at  $r = 3$  points the differential form is always logarithmic.

The deformation datum  $(Z, \omega)$  associated to  $g$  satisfies the following compatibilities with the tail covers. We refer to [20], Proposition 1.8 and (2) for proofs of these statements. For  $i \in \mathbb{B}$ , we let  $h_i$  (resp.  $pm_i$ ) be the conductor (resp. ramification index) of a wild ramification point of the corresponding tail cover of type  $C_i$ , as defined above. We put  $\sigma_i = h_i/m_i$ . We also use the convention  $\sigma_i = 0$  for  $i \notin \mathbb{B}$ .

- (a) If  $C_i$  is the conjugacy class of a  $p$ -cycle then  $\bar{Q}_i$  is unbranched in  $\bar{Z}_0 \rightarrow \bar{X}_0$  and  $\omega$  has a simple pole at all points of  $\bar{Z}_0$  above  $\bar{Q}_i$ .
- (b) Otherwise,  $\bar{Z}_0 \rightarrow \bar{X}_0$  is branched of order  $m_i$  at  $\bar{Q}_i$ , and  $\omega$  has a zero of order  $h_i - 1$  at the points of  $\bar{Z}_0$  above  $\bar{Q}_i$ .
- (c) The map  $\bar{Z}_0 \rightarrow \bar{X}_0$  is unbranched outside  $\{\bar{Q}_i\}$ . All poles and zeros of  $\omega$  are above the  $\bar{Q}_i$ .
- (d) The invariants  $\sigma_i$  satisfy  $\sum_{i \in \mathbb{B}} \sigma_i = r - 2$ .

The set  $(\sigma_i)$  is called the *signature* of the deformation datum  $(Z, \omega)$ .

**Proposition 4.4.** *Suppose that  $r = 3, 4$ . We fix rational numbers  $(\sigma_1, \dots, \sigma_r)$  with  $\sigma_i \in \frac{1}{p-1}\mathbb{Z}$  and  $0 \leq \sigma_i < 1$ , and  $\sum_{i=1}^r \sigma_i = r - 2$ . We fix points  $\bar{Q}_1 = 0, \bar{Q}_2 = 1, \dots, \bar{Q}_r = \infty$  on  $\bar{X}_0 \simeq \mathbb{P}_k^1$ . Then there exists a deformation datum of signature  $(\sigma_i)$ , unique up to scaling. If further the  $\bar{Q}_i$  are general, the deformation datum is logarithmic and unique up to isomorphism.*

*Proof.* In the case that  $r = 3$  this is proved in [20]. (The proof in this case is similar to the proof in the case that  $r = 4$  which we give below.) Suppose that  $r = 4$ . Let  $\mathbb{B} = \{1 \leq i \leq r \mid \sigma_i \neq 0\}$ . We write  $\bar{Q}_3 = \lambda \in \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$  and  $\sigma_i = a_i/(p-1)$ . (If  $\omega$  is the deformation datum associated with  $\bar{g}$ , then  $a_i = h_i(p-1)/m_i$ .)

It is shown in [3], Chapter 3, that a deformation datum of signature  $(\sigma_i)$  consists of a differential form  $\omega$  on the cover  $\bar{Z}_0$  of  $\bar{X}_0$  defined as a connected component of the (normalization of the) projective curve with Kummer equation

$$(4.3) \quad z^{p-1} = x^{a_1}(x-1)^{a_2}(x-\lambda)^{a_3}.$$

The degree of  $\bar{Z}_0 \rightarrow \bar{X}_0$  is

$$m := \frac{p-1}{\gcd(p-1, a_1, a_2, a_3, a_4)}.$$

The differential form  $\omega$  may be written as

$$(4.4) \quad \omega = \epsilon \frac{z \, dx}{x(x-1)(x-\lambda)} = \epsilon \frac{x^{p-a_1}(x-1)^{p-1-a_2}(x-\lambda)^{p-1-a_3} z^p \, dx}{x^p(x-1)^p(x-\lambda)^p} \frac{dx}{x},$$

where  $\epsilon \in k^\times$  is a unit.

To show the existence of the deformation datum, it suffices to show that one may choose  $\epsilon$  such that  $\omega$  is logarithmic or exact, or, equivalently, such that  $\omega$  is fixed or killed by the Cartier operator  $\mathcal{C}$ . It follows from standard properties of the Cartier operator, (4.4), and the assumption that  $a_1 + a_2 + a_3 + a_4 = 2(p-1)$  that  $\mathcal{C}\omega = c^{1/p} \epsilon^{(1-p)/p} \omega$ , where

$$(4.5) \quad c = \sum_{j=\max(0, p-1-a_2-a_4)}^{\min(a_4, p-1-a_3)} \binom{p-1-a_2}{a_4-j} \binom{p-1-a_3}{j} \lambda^j.$$

Note that  $c$  is the coefficient of  $x^p$  in  $x^{p-a_1}(x-1)^{p-1-a_2}(x-\lambda)^{p-1-a_3}$ . One easily checks that  $c$  is nonzero as polynomial in  $\lambda$ . It follows that  $\omega$  defines an exact

differential form if and only if  $\lambda$  is a zero of the polynomial  $c$ . This does not happen if  $\{0, 1, \lambda, \infty\}$  is general.

We assume that  $c(\lambda) \neq 0$ . Since  $k$  is algebraically closed, we may choose  $\epsilon \in k^\times$  such that  $\epsilon^{p-1} = c$ . Then  $\mathcal{C}\omega = \omega$ , and  $\omega$  defines a logarithmic deformation datum. It is easy to see that  $\omega$  is unique, up to multiplication by an element of  $\mathbb{F}_p^\times$ .  $\square$

## 5. THE TAIL COVERS

In Section 4, we have seen that associated with a Galois cover with bad reduction is a set of (primitive) tail covers. In this section, we analyze the possible tail covers for conjugacy classes  $e \neq p$  and  $e_1$ - $e_2$  of  $S_p$ . Recall from Section 2 that these are conjugacy classes which occur in the 3-point covers obtained as degeneration of the pure-cycle 4-point covers.

The following lemma shows the existence of primitive tail covers for the conjugacy classes occurring in the degeneration of single-cycle 4-point covers (Theorem 2.3).

**Lemma 5.1.** (a) *Let  $2 \leq e < p-1$  be an integer. There exists a primitive tail cover  $\varphi_e : T_e \rightarrow \mathbb{P}_k^1$  of ramification type  $e$ . Its Galois group is  $A_p$  if  $e$  is odd and  $S_p$  if  $e$  is even. The wild branch point of  $\varphi_e$  has inertia group of order  $p(p-1)/\gcd(p-1, e-1) =: pm_e$ . The conductor is  $h_e := (p-e)/\gcd(p-1, e-1)$ .*

(b) *In the case that  $e = p-1$ , there exists a primitive tail cover  $\varphi_e$  of ramification type  $e$ , with Galois group  $\mathbb{F}_p \rtimes \mathbb{F}_p^*$ . The cover is totally branched above the wild branch point and has conductor  $h_{p-1} = 1$ .*

(c) *Let  $2 \leq e_1 \leq e_2 \leq p-1$  be integers with  $e_1 + e_2 \leq p$ . There is a primitive tail cover  $\varphi_{e_1, e_2} : T_{e_1, e_2} \rightarrow \mathbb{P}_k^1$  of ramification type  $e_1$ - $e_2$ . The wild branch point of  $\varphi_{e_1, e_2}$  has inertia group of order  $p(p-1)/\gcd(p-1, e_1+e_2-2) =: pm_{e_1, e_2}$ . The conductor is  $h_{e_1, e_2} := (p+1-e_1-e_2)/\gcd(p-1, e_1+e_2-2)$ .*

*In all three cases, the tail cover is unique with the given ramification when considered as a mere cover.*

*Proof.* Let  $2 \leq e \leq p-1$  be an integer. We define the primitive tail cover  $\varphi_e$  as the Galois closure of the degree- $p$  cover  $\bar{\varphi}_e : \bar{T}_e := \mathbb{P}^1 \rightarrow \mathbb{P}^1$  given by

$$(5.1) \quad y^p + y^e = x, \quad (x, y) \mapsto x.$$

One easily checks that this is the unique degree- $p$  cover between projective lines with one wild branch point and the required tame ramification.

The decomposition group  $G_e$  of  $T_e$  is contained in  $S_p$ . We note that the normalizer in  $S_p$  of a Sylow  $p$ -subgroup has trivial center. Therefore the inertia group  $I$  of a wild ramification point of  $\varphi_e$  is contained in  $\mathbb{F}_p \rtimes_\chi \mathbb{F}_p^*$ , where  $\chi : \mathbb{F}_p^* \rightarrow \mathbb{F}_p^*$  is an injective character. Therefore it follows from [6], Proposition 2.2.(i) that  $\gcd(h_e, m_e) = 1$ . The statement on the wild ramification follows now directly from the Riemann–Hurwitz formula (as in [20], Lemma 4.10). Suppose that  $e$  is odd. Then  $m_e = (p-1)/\gcd(p-1, e-1)$  divides  $(p-1)/2$ . Therefore in this case both the tame and the wild ramification groups are contained in  $A_p$ . This implies that the Galois group  $G_e$  of  $\varphi_e$  is a subgroup of  $A_p$ .

To prove (a), we suppose that  $e \neq p-1$ . We show that the Galois group  $G_e$  of  $\varphi_e$  is  $A_p$  or  $S_p$ . Suppose that this is not the case. Proposition 3.1 implies that  $e = p-2 = 2^r - 1$ . Moreover,  $G_e$  is a subgroup of  $\mathrm{P}\Gamma\mathrm{L}_2(2^r) \simeq \mathrm{PSL}_2(2^r) \rtimes \mathbb{Z}/r\mathbb{Z}$ . The normalizer in  $\mathrm{P}\Gamma\mathrm{L}_2(2^r)$  of a Sylow  $p$ -subgroup  $I$  is  $\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/2r\mathbb{Z}$ . The computation of the wild ramification shows that the inertia group  $I(\eta)$  of the wild ramification

point  $\eta$  is isomorphic to  $\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/\frac{p-1}{2}\mathbb{Z}$ . Therefore  $\text{P}\Gamma\text{L}_2(2^r)$  contains a subgroup isomorphic to  $I(\eta)$  if and only if  $p = 17 = 2^4 + 1$ , in which case  $I(\eta) = N_{\text{P}\Gamma\text{L}_2(2^r)}(I)$ . We conclude that if  $G_e \not\cong S_p, A_p$  then  $e = 15$  and  $p = 17$ . However, in this last case one may check using Magma that a suitable specialization of (5.1) has Galois group  $A_{17}$ . As before, we conclude that  $G_e \simeq A_{17}$ .

Now suppose that  $e = p - 1$ . It is easy to see that the Galois closure of  $\bar{\varphi}_{p-1}$  is in this case the cover  $\varphi_{p-1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  obtained by dividing out  $\mathbb{F}_p \rtimes \mathbb{F}_p^* \subset \text{PGL}_2(p) = \text{Aut}(\mathbb{P}^1)$ . This proves (b).

Let  $e_1, e_2$  be as in the statement of (c). As in the proof of (a), we define  $\varphi_{e_1, e_2}$  as the Galois closure of a non-Galois cover  $\bar{\varphi}_{e_1, e_2} : \bar{T}_{e_1, e_2} \rightarrow \mathbb{P}^1$  of degree  $p$ . The cover  $\bar{\varphi}_{e_1, e_2}$ , if it exists, is given by an equation

$$(5.2) \quad F(y) := y^{e_1}(y-1)^{e_2}\tilde{F}(y) = x, \quad (x, y) \mapsto x,$$

where  $\tilde{F}(y) = \sum_{i=0}^{p-e_1-e_2} c_i y^i$  has degree  $p - e_1 - e_2$ . We may assume that  $c_{p-e_1-e_2} = 1$ . The condition that  $\bar{\varphi}_{e_1, e_2}$  has exactly three ramification points  $y = 0, 1, \infty$  yields the condition  $F'(y) = \gamma t^{e_1-1}(t-1)^{e_2-1}$ . Therefore the coefficients of  $\tilde{F}$  satisfy the recursion

$$(5.3) \quad c_i = c_{i-1} \frac{e_1 + e_2 + i - 1}{e_1 + i}, \quad i = 1, \dots, p - e_1 - e_2.$$

This implies that the  $c_i$  are uniquely determined by  $c_{p-e_1-e_2} = 1$ . Conversely, it follows that the polynomial  $F$  defined by these  $c_i$  has the required tame ramification. The statement on the wild ramification follows from the Riemann–Hurwitz formula, as in the proof of (a).  $\square$

It remains to analyze the number of tail covers, and their automorphism groups. Due to the nature of our argument, we will only need to carry out this analysis for the tails of ramification type  $e$ . From Lemma 5.1, it follows already that the map  $\varphi_C : T_C \rightarrow \mathbb{P}^1$  is unique. However, part of the datum of a tail cover is an isomorphism  $\alpha : \text{Gal}(T_C, \mathbb{P}^1) \xrightarrow{\sim} G_C$ . For every  $\tau \in \text{Aut}(G_C)$ , the tuple  $(\varphi, \tau \circ \alpha)$  also defines a tail cover, which is potentially non-equivalent. Modification by  $\tau$  leaves the cover unchanged as a  $G_C$ -Galois cover if and only if  $\tau \in \text{Inn}(G_C)$ . However, the weaker notion of equivalence for tail covers implies that  $\tau$  leaves the cover unchanged as a tail cover if and only if  $\tau$  can be described as conjugation by an element of  $N_{\text{Aut}(T)}(G_C)$ . Thus, the number of distinct tail covers corresponding to a given mere cover is the order of the cokernel of the map

$$N_{\text{Aut}(T_C)}(G_C) \rightarrow \text{Aut}(G_C)$$

given by conjugation. Denote by  $\text{Aut}_{G_C}(\varphi_C)$  the kernel of this map, or equivalently the set of  $G_C$ -equivariant automorphisms of  $T_C$ . It follows finally that the number of tail covers corresponding to  $\varphi_C$  is

$$(5.4) \quad \frac{|\text{Aut}(G_C)| |\text{Aut}_{G_C}(\varphi_C)|}{|N_{\text{Aut}(T_C)}(G_C)|}.$$

Finally, denote by  $\text{Aut}_{G_C}^0(\varphi_C) \subset \text{Aut}_{G_C}(\varphi_C)$  the subset of automorphisms which fix the chosen ramification point  $\eta$ . We now simultaneously compute these automorphism groups and show that in the single-cycle case, we have a unique tail cover.

**Lemma 5.2.** *Let  $2 \leq e \leq p - 1$  be an integer.*

- (a) The group  $\text{Aut}_{G_e}(\varphi_e)$  (resp.  $\text{Aut}_{G_e}^0(\varphi_e)$ ) is cyclic of order  $(p-e)/2$  (resp.  $h_e$ ) if  $e$  is odd and  $p-e$  (resp.  $h_e$ ) is  $e$  is even.  
 (b) There is a unique primitive tail cover of type  $e$ .

*Proof.* First note that the definition of  $\text{Aut}_{G_e}(\varphi_e)$  implies that any element induces an automorphism of any intermediate cover of  $\varphi_e$ , and in particular induces automorphisms of  $\bar{T}_e$  and  $\mathbb{P}^1$ . Choose a primitive  $(p-e)$ th root of unity  $\zeta \in \bar{\mathbb{F}}_p$ . Then  $\mu(x, y) = (\zeta^e x, \zeta y)$  is an automorphism of  $\bar{T}_e$ . One easily checks that  $\mu$  generates the group of automorphisms of  $\bar{T}_e$  which induces automorphisms of  $\mathbb{P}^1$  under  $\varphi_e$ , and that furthermore  $T_e$  is Galois over  $\mathbb{P}^1/\langle\mu\rangle$ , so in particular every element of  $\mu$  lifts to an automorphism of  $T_e$ . Taking the quotient by the action of  $\mu$ , we obtain a diagram

$$(5.5) \quad \begin{array}{ccc} \bar{T}_e & \longrightarrow & \bar{T}'_e = \bar{T}_e/\langle\mu\rangle \\ \bar{\varphi}_e \downarrow & & \downarrow \bar{\psi}_e \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1/\langle\mu\rangle. \end{array}$$

Since we know the ramification of the other three maps, one easily computes that the tame ramification of  $\bar{\psi}_e$  is  $e-(p-e)$ . Let  $\psi_e : T'_e \rightarrow \mathbb{P}^1$  be the Galois closure of  $\bar{\psi}_e$ .

We now specialize to the case that  $e$  is odd. Since  $G_e = A_p$  does not contain an element of cycle type  $e-(p-e)$ , it follows that the Galois group  $G'$  of  $\psi_e$  is  $S_p$ . Therefore it follows by degree considerations that the cover  $T_e \rightarrow T'_e$  is cyclic of degree  $(p-e)/2$ . Denote by  $Q$  the Galois group of the cover  $T_e \rightarrow \mathbb{P}^1/\langle\mu\rangle$ . This is a group of order  $p!(p-e)/2$ , which contains normal subgroups isomorphic to  $A_p$  and  $\mathbb{Z}/\frac{p-e}{2}\mathbb{Z}$ , respectively. It follows that  $Q = \mathbb{Z}/\frac{p-e}{2}\mathbb{Z} \rtimes S_p$ . Note that  $\text{Aut}_{G_e}(\varphi_e)$  is necessarily a subgroup of  $Q$ . In fact, it is precisely the subgroup of  $Q$  which commutes with every element of  $A_p \subseteq Q$ . One easily checks that the semidirect product cannot be split, and that  $\text{Aut}_{G_e}(\varphi_e)$  is precisely the normal subgroup  $\mathbb{Z}/\frac{p-e}{2}\mathbb{Z}$ , that is the Galois group of  $T_e$  over  $T'_e$ .

To compute  $\text{Aut}_{G_e}^0(\varphi_e)$  we need to compute the order of the inertia group of a wild ramification point of  $T_e$  in the map  $T_e \rightarrow T'_e$ . Since a wild ramification point of  $T'_e$  has inertia group of order  $p(p-1) = pm_e \gcd(p-1, e-1)$ , we know the orders of the inertia groups of three of the four maps, and conclude that  $\text{Aut}_{G_e}^0(\varphi_e)$  has order  $h_e = (p-e)/\gcd(p-1, e-1)$ . This proves (a) in the case  $e$  is odd.

For (b), we simply observe that since  $Q \subset N_{\text{Aut}(T_e)}(G_e)$ , we have

$$\frac{|\text{Aut}(G_e)| |\text{Aut}_{G_e}(\varphi_e)|}{|N_{\text{Aut}(T_e)}(G_e)|} \leq \frac{p! \frac{p-e}{2}}{|Q|} = 1,$$

so the tail cover is unique, as desired.

We now treat the case that  $e$  is even. For (a), if  $e < p-1$ , the Galois group of  $\bar{\psi}_e$  is equal to the Galois group of  $\bar{\varphi}_e$ , which is isomorphic to  $S_p$ . We conclude that the degree of  $T_e \rightarrow T'_e$  is  $p-e$  in this case, and the group  $Q$  defined as above is a direct product  $\mathbb{Z}/(p-e)\mathbb{Z} \times S_p$ . Similarly to the case that  $e$  is odd, we conclude that  $\text{Aut}_{G_e}(\varphi_e)$  (resp.  $\text{Aut}_{G_e}^0(\varphi_e)$ ) is cyclic of order  $p-e$  (resp.  $h_e$ ) in this case, as desired. On the other hand, if  $e = p-1$ , we have that  $p-e = 1$ , hence  $\mu$  is trivial, and we again conclude that (a) holds. Finally, (b) is trivial: if  $e < p-1$ , the Galois group of  $\varphi_e$  is  $S_p$  and  $\text{Aut}(S_p) = S_p$ . Therefore there is a unique tail cover in this



case. The same conclusion holds in the case that  $e = p - 1$ , since  $G_{p-1} \simeq \mathbb{F}_p \rtimes_{\chi} \mathbb{F}_p^*$  and  $\text{Aut}(G_{p-1}) = G_{p-1}$ . The statement of the lemma follows.  $\square$

*Remark 5.3.* In the case of  $e_1$ - $e_2$  tail covers, there may in fact be more than one structure on a given mere cover. However, we will not need to know this number for our argument.

## 6. REDUCTION OF 3-POINT COVERS

In this section, we (almost) compute the number of 3-point covers with bad reduction for ramification types  $(p; e_1, e_2, e_3, e_4)$ . More precisely, we compute this number in the case that not both  $e_3$  and  $e_1 + e_2$  are even. In the remaining case, we only compute this number up to a factor 2, which is good enough for our purposes. Although we restrict to types of the above form, our strategy applies somewhat more generally. The results of this section rely on the results of Wewers [20], who gives a precise formula for the number of lifts of a given special  $G$ -map (Section 4) in the 3-point case.

We fix a type  $\tau = (p; e_1, e_2, e_3, e_4)$  satisfying the genus-0 condition  $\sum_i e_i = 2p + 2$ . We allow  $e_3$  or  $e_4$  to be  $p$ , although this is not the case that ultimately interests us; see below for an explanation. We do however assume throughout that we are not in the exceptional case  $\tau = (5; 2, 2, 4, 4)$ . According to Lemma 5.1, we may fix a set of primitive tail covers  $\bar{g}_i$  of type  $C_i$ , for  $i$  such that  $C_i \neq p$ . Moreover, by Proposition 4.4 we have a (unique) deformation datum, so we know there exists at least one special  $G$ -map  $\bar{g}$  of type  $\tau$ . Lemma 5.2 implies moreover that the number of special  $G$ -maps is equal to the number of  $e_1$ - $e_2$  tail covers. Wewers ([20], Theorem 3) shows that there exists a  $G$ -Galois cover  $g : Y \rightarrow \mathbb{P}^1$  in characteristic zero with bad reduction to characteristic  $p$ , and more specifically with stable reduction equal to the given special  $G$ -map  $\bar{g}$ . Moreover, Wewers gives a formula for the number  $\tilde{L}(\bar{g})$  for lifts of the given special  $G$ -map  $\bar{g}$ .

In order to state his formula, we need to introduce one more invariant. Let  $\text{Aut}_G^0(\bar{g})$  be the group of  $G$ -equivariant automorphisms of  $\bar{Y}$  which induce the identity on the original component  $\bar{X}_0$ . Choose  $\gamma \in \text{Aut}_G^0(\bar{g})$ , and consider the restriction of  $\gamma$  to the original component  $\bar{X}_0$ . Let  $\bar{Y}_0$  be an irreducible component of  $\bar{Y}$  above  $\bar{X}_0$  whose inertia group is the fixed Sylow  $p$ -subgroup  $I$  of  $G$ . As in (4.2), we write  $G_0 = I \rtimes_{\chi} H_0 \subset \mathbb{F}_p \rtimes_{\chi} \mathbb{F}_p^*$  for the decomposition group of  $\bar{Y}_0$ . Wewers ([20], proof of Lemma 2.17) shows that  $\gamma_0 := \gamma|_{\bar{Y}_0}$  centralizes  $H_0$  and normalizes  $I$ , i.e.  $\gamma_0 \in C_{N_G(I)}(H_0)$ . Since  $\bar{Y}|_{\bar{X}_0} = \text{Ind}_{G_0}^G \bar{Y}$  and  $\gamma$  is  $G$ -equivariant, it follows that the restriction of  $\gamma$  to  $\bar{X}_0$  is uniquely determined by  $\gamma_0$ . We denote by  $n'(\tau)$  the order of the subgroup consisting of those  $\gamma_0 \in C_{N_G(I)}(H_0)$  such that there exists a  $\gamma \in \text{Aut}_G^0(\bar{g})$  with  $\gamma|_{\bar{Y}_0} = \gamma_0$ . Our notation is justified by Corollary 6.3 below.

Wewers ([20], Corollary 4.8) shows that

$$(6.1) \quad |\tilde{L}(\bar{g})| = \frac{p-1}{n'(\tau)} \prod_{i \in \mathbb{B}} \frac{h_{C_i}}{|\text{Aut}_{G_{C_i}}^0(\bar{g}_{C_i})|}.$$

The numbers are as defined in Section 4. (Note that the group  $\text{Aut}_{G_{C_i}}^0(\bar{g}_{C_i})$  is defined differently from the group  $\text{Aut}_G^0(\bar{g})$ .)

To compute the number of curves with bad reduction, we need to compute the number  $n'(\tau)$  defined above. As explained by Wewers ([20, Lemma 2.17]), one may express the number  $n'(\tau)$  in terms of certain groups of automorphisms of the tail

covers. However, there is a mistake in the concrete description he gives of  $\text{Aut}_G^0(\bar{g})$  in terms of the tails, therefore we do not use Wewers' description. For a corrected version, we refer to the manuscript [17].

The difficulty we face in using Wewers' formula directly is that we do not know the Galois group  $G_{e_1-e_2}$  of the  $e_1-e_2$  tail. This prevents us from directly computing both the number of  $e_1-e_2$  tails, and the term  $n'(\tau)$ . We avoid this problem by using the following trick. We first consider covers of type  $\tau^* = (p; e_1-e_2, \varepsilon, p)$ , with  $\varepsilon = p+2-e_1-e_2$ , which all have bad reduction. This observation lets us compute  $n'(\tau^*)$  from Wewers' formula. We then show that for covers of type  $\tau = (p; e_1-e_2, e_3, e_4)$ , the number  $n'(\tau)$  essentially only depends on  $e_1$  and  $e_2$ , allowing us to compute  $n'(\tau)$  from  $n'(\tau^*)$ . A problem with this method is that in the case that the Galois groups of covers with type  $\tau$  and  $\tau^*$  are not equal, the numbers  $n'(\tau)$  and  $n'(\tau^*)$  may differ by a factor 2. Therefore in this case, we are able to determine the number of covers of type  $\tau$  with bad reduction only up to a factor 2.

In Lemma 2.2, we have counted non-Galois covers, but in this section, we deal with Galois covers. Let  $G(\tau)$  be the Galois group of a cover of type  $\tau$ . This group is well-defined and either  $A_p$  or  $S_p$ , by Corollary 3.2. We write  $\gamma(\tau)$  for the quotient of the number of Galois covers of type  $\tau$  by the Hurwitz number  $h(\tau)$ . By Lemma 3.3, it follows that  $\gamma(\tau)$  is 2 if  $G$  is  $A_p$  and 1 if it is  $S_p$ . The number  $\gamma(\tau)$  will drop out from the formulas as soon as we pass back to the non-Galois situation in Section 7.

We first compute the number  $n'(\tau^*)$ . We note that by Corollary 3.2, the Galois group  $G(\tau^*)$  of a cover of type  $\tau^*$  is  $A_p$  if  $e_1 + e_2$  is even and  $S_p$  otherwise. In particular, we see that  $G(\tau) = G(\tau^*)$  unless  $e_1 + e_2$  and  $e_3$  are both even. In this case we have that  $G(\tau) = S_p$  and  $G(\tau^*) = A_p$ . Recall from Lemma 5.2 that there is a unique tail cover for the single-cycle tails. We denote by  $N_{e_1-e_2}$  the number of  $e_1-e_2$  tails, and by  $\text{Aut}_{e_1-e_2}^0$  the group  $\text{Aut}_{G_{e_1-e_2}}^0(\bar{g}_{e_1-e_2})$  for any tail cover  $\bar{g}_{e_1-e_2}$  as in Lemma 5.1. Note that since  $\bar{g}_{e_1-e_2}$  is unique as a mere cover, and the definition of  $\text{Aut}_{G_{e_1-e_2}}^0(\bar{g}_{e_1-e_2})$  is independent of the  $G$ -structure, this notation is well-defined. We similarly have from (6.1) that  $|\tilde{L}(\bar{g})|$  depends only on  $\tau$ , so we write  $\tilde{L}(\tau) := |\tilde{L}(\bar{g})|$  for any special  $G$ -map  $\bar{g}$  of type  $\tau$ .

**Lemma 6.1.** *Let  $\tau^*$  be as above. Then*

$$n'(\tau^*) = \frac{(1 + \delta_{e_1, e_2})N_{e_1-e_2}(p-1)}{\gcd(p-1, e_1+e_2-2)\gamma(\tau^*)|\text{Aut}_{e_1-e_2}^0|}.$$

Here  $\delta_{e_1, e_2}$  is the Kronecker  $\delta$ .

*Proof.* Lemma 2.2 implies that the Hurwitz number  $h(\tau^*)$  equals  $(p+1-e_1-e_2)/2$  if  $e_1 = e_2$  and  $(p+1-e_1-e_2)$  otherwise. Since all covers of type  $\tau^*$  have bad reduction,  $h(\tau^*)\gamma(\tau^*)$  is equal to  $N_{e_1-e_2} \cdot \tilde{L}(\tau^*)$ . The statement of the lemma follows by applying Lemmas 5.1(c), 5.2, and (6.1).  $\square$

We now analyze  $n'$  in earnest. For convenience, for  $i \in \mathbb{B}$  we also introduce the notation  $\widetilde{\text{Aut}}_{G_i}(\bar{g}_i)$  for the group of  $G$ -equivariant automorphisms of the induced tail cover  $\text{Ind}_{G_i}^G(\bar{g}_i)$ . Recall also that  $\xi_i$  is the node connecting  $\bar{X}_0$  to  $\bar{X}_i$ . We note that  $n'$  may be analyzed tail by tail, in the sense that given  $\gamma_0 \in C_{NG(I)}(H_0)$ , we have that  $\gamma_0$  lifts to  $\text{Aut}^0(\bar{g})$  if and only if for each  $i \in \mathbb{B}$ , there is some  $\gamma_i \in \widetilde{\text{Aut}}_{G_i}(\bar{g}_i)$  whose action on  $\bar{g}_i^{-1}(\xi_i)$  is compatible with  $\gamma_0$ . The basic proposition underlying the behavior of  $n'$  is then the following:

**Proposition 6.2.** *Suppose  $G = S_p$  or  $A_p$ , and we have a special  $G$ -map  $\bar{g} : \bar{Y} \rightarrow \bar{X}$ . Then:*

- (a) *For  $i \in \mathbb{B}$ , the  $G$ -equivariant automorphisms of  $\bar{g}^{-1}(\xi_i)$  form a cyclic group.*
- (b) *Given an element  $\gamma_0 \in C_{N_G(I)}(H_0)$  and  $i \in \mathbb{B}$ , there exists  $\gamma_i \in \widetilde{\text{Aut}}_{G_i}(\bar{g}_i)$  agreeing with the action of  $\gamma_0$  on  $\bar{g}^{-1}(\xi_i)$  if and only if there exists  $\gamma'_i \in \widetilde{\text{Aut}}_{G_i}(\bar{g}_i)$  having the same orbit length on  $\bar{g}^{-1}(\xi_i)$  as  $\gamma_0$  has.*

*Proof.* For (a), if  $\tilde{\xi}_i$  is a point above  $\xi_i$  lying on the chosen component  $\bar{Y}_0$ , one easily checks that a  $G$ -equivariant automorphism  $\gamma$  of  $\bar{g}^{-1}(\xi_i)$  is determined by where it sends  $\tilde{\xi}_i$ , which can in turn be represented by an element  $g \in G$  chosen so that  $g(\tilde{\xi}_i) = \gamma(\tilde{\xi}_i)$ . Note that  $\gamma \neq g$ ; in fact, if  $\gamma, \gamma'$  are determined by  $g, g'$ , the composition law is that  $\gamma \circ \gamma'$  corresponds to  $g'g$ . Such a  $g$  yields a choice of  $\gamma$  if and only if we have the equality of stabilizers  $G_{\tilde{\xi}_i} = G_{g(\tilde{\xi}_i)}$ . Now, any  $h \in G_{\tilde{\xi}_i}$  is necessarily in  $G_0$ , and using that  $I \subseteq G_{\tilde{\xi}_i}$ , we find that we must have  $gIg^{-1} \subseteq G_0$ . But  $I$  contains the only  $p$ -cycles in  $G_0$ , so we conclude  $g \in N_G I$ . However, since  $I$  fixed  $\tilde{\xi}_i$ , the choices of  $G$  may be taken modulo  $I$ , so we conclude that they lie in  $N_G I / I$ . Finally, since  $G = S_p$  or  $A_p$ , we have that  $N_G I / I$  is cyclic, isomorphic to  $\mathbb{Z}/(p-1)\mathbb{Z}$  if  $G = S_p$  and to  $\mathbb{Z}/(\frac{p-1}{2})\mathbb{Z}$  if  $G = A_p$ .

(b) then follows immediately, since the actions of both  $\gamma_0$  and  $\gamma'_i$  on  $\bar{g}^{-1}(\xi_i)$  lie in the same cyclic group; we can take  $\gamma_i$  to be an appropriate power of  $\gamma'_i$ .  $\square$

**Corollary 6.3.** *For  $\tau$  as above,  $n'(\tau)$  is well defined.*

*Proof.* We know that  $G = S_p$  or  $A_p$ , and we also know by Proposition 4.4 and Lemma 5.1 that the deformation datum is uniquely determined, and so are the tail covers, at least as mere covers. But the description of  $n'(\tau)$  given by Proposition 6.2 is visibly independent of the  $G$ -structure on the tail covers, so we obtain the desired statement.  $\square$

We can now obtain the desired comparison of  $n'(\tau)$  with  $n'(\tau^*)$ .

**Proposition 6.4.** *Let  $\tau = (p; e_1 - e_2, e_3, e_4)$  be a type satisfying the genus-0 condition, and let  $\tau^*$  be the corresponding modified type. Then if  $G(\tau) = G(\tau^*)$  we have  $n'(\tau) = n'(\tau^*)$ . Otherwise,  $n'(\tau) \in \{2n'(\tau^*), n'(\tau^*)\}$ .*

*Proof.* Let  $\gamma_0$  be a generator of  $C_{N_G(I)}(H_0)$ . We ask which powers of  $\gamma_0$  extend to an element of  $\text{Aut}_G^0(\bar{g})$ , and we analyze this question tail by tail. Fix a tail  $\bar{X}_i$ , and suppose that it is a single-cycle tail of length  $e := e_i$ . The crucial assertion is that  $\gamma_0$  itself (and hence all its powers) always extends to  $\bar{X}_i$ .

First suppose that  $e < p - 1$  is even. Thus  $G = G_i = S_p$ , and  $\widetilde{\text{Aut}}_G(\bar{g}_i) = \text{Aut}_{G_i}(\bar{g}_i)$ . Now,  $\gamma_0$  acts on the fiber of  $\xi_i$  with orbit length  $(p-1)/m_e = \gcd(p-1, p-e)$ . On the other hand, by Lemma 5.1 we have that  $h_e = (p-e)/\gcd(p-1, e-1)$ . Lemma 5.2 implies that if  $\gamma_i \in \text{Aut}_{G_i}(\bar{g}_i)$  is a generator, then the order of  $\gamma_i$  is  $p-e$ , and also that  $\text{Aut}_{G_i}^0(\bar{g}_i)$  has order  $h_e$ . We conclude that an orbit of  $\gamma_i$  has length  $\gcd(p-1, e-1) = \gcd(p-1, p-e)$ , and thus by Proposition 6.2 that  $\gamma_0$  extends to  $\bar{X}_i$ , as claimed.

The next case is that  $e$  is odd, and  $G = A_p$ . This proceeds exactly as before, except that both orbits in question have length  $\gcd(p-1, p-e)/2$ . Now, suppose  $e$  is odd, but  $G = S_p$ . Then the orbit length of  $\gamma_0$  is  $\gcd(p-1, p-e)$ . We have  $\widetilde{\text{Aut}}_G(\bar{g}_i)$  equal to the  $G$ -equivariant automorphisms of  $\text{Ind}_{A_p}^{S_p}(\bar{g}_i)$ . These contain

induced copies of the  $G$ -equivariant automorphisms of  $\bar{g}_i$ , so in particular we know we have elements of orbit length  $\gcd(p-1, e-1)/2$ . However, in fact one also has a  $G$ -equivariant automorphism exchanging the two copies of  $\bar{g}_i$ , and whose square is the generator of the  $A_p$ -equivariant automorphisms of  $\bar{g}_i$ . One may think of this as coming from the automorphism constructing in Lemma 5.2 inducing the isomorphism between the two different  $A_p$ -structures on the tail cover. We thus have an element of  $\widetilde{\text{Aut}}_G(\bar{g}_i)$  of orbit length  $\gcd(p-1, e-1)$ , and  $\gamma_0$  extends to the tail in this case as well.

Finally, if  $e = p-1$  then  $m_i = p-1$  and thus  $\gamma_0$  acts as the identity on the fiber of  $\xi_i$ . The claim is trivially satisfied in this case.

It follows that extending  $\gamma_0$  to the  $e$ -tails imposes no condition when  $e < p$ , and of course we do not have tails in the case that  $e = p$ . Therefore the only non-trivial condition imposed in extending  $\gamma_0$  is the extension to the  $e_1$ - $e_2$ -tail.

In the case that  $G(\tau) = G(\tau^*)$  we conclude the desired statement from Proposition 6.2, since the orbit lengths in question are clearly the same in both cases. Suppose that  $G(\tau) \neq G(\tau^*)$ . This happens if and only if both  $e_1 + e_2$  and  $e_3$  are even. In this case we have that  $G(\tau) = S_p$  and  $G(\tau^*) = A_p$ . Here, we necessarily have that  $e_1 + e_2, e_3, e_4$  are all even, so the only conditions imposed on either  $n'(\tau)$  or  $n'(\tau^*)$  come from the  $e_1$ - $e_2$  tail. Since the orbit of  $\gamma_0$  is twice as long in the case of  $\tau$ , the answers can differ by at most a factor of 2 in this case, as desired.  $\square$

Let  $2 \leq e_1 \leq e_2 \leq e_3 \leq e_4 < p$  be integers with  $\sum_i e_i = 2p+2$  and  $e_1 + e_2 \leq p$ . The following corollary translates Proposition 6.4 into an estimate for the number of Galois covers of type  $\tau = (p; e_1-e_2, e_3, e_4)$  with bad reduction. Theorem 1.2 is a special case.

**Corollary 6.5.** *Let  $\tau = (p; e_1-e_2, e_3, e_4)$  with  $\tau \neq (5; 2-2, 4, 4)$ . The number of mere covers of type  $\tau$  with bad reduction to characteristic  $p$  is equal to*

$$\begin{cases} \delta(\tau)(p+1-e_1-e_2) & \text{if } e_1 \neq e_2, \\ \delta(\tau)(p+1-e_1-e_2)/2 & \text{if } e_1 = e_2, \end{cases}$$

where  $\delta(\tau) \in \{1, 2\}$ , and  $\delta = 1$  unless  $e_1 + e_2$  and  $e_3$  are both even.

*Proof.* We recall that the number of Galois covers of type  $\tau$  with bad reduction is equal to  $N_{e_1-e_2} \cdot \tilde{L}(\tau^*)$ . It follows from Lemma 5.1.(c) and (6.1) that this number is

$$\begin{cases} \frac{\gamma(\tau^*)n'(\tau^*)}{n'(\tau)}(p+1-e_1-e_2) & \text{if } e_1 \neq e_2, \\ \frac{\gamma(\tau^*)n'(\tau^*)}{n'(\tau)}(p+1-e_1-e_2)/2 & \text{if } e_1 = e_2. \end{cases}$$

The definition of the Galois factor  $\gamma(\tau)$  implies that the number of mere covers of type  $\tau$  with bad reduction is

$$\begin{cases} \frac{\gamma(\tau^*)}{\gamma(\tau)} \frac{n'(\tau^*)}{n'(\tau)}(p+1-e_1-e_2) & \text{if } e_1 \neq e_2, \\ \frac{\gamma(\tau^*)}{\gamma(\tau)} \frac{n'(\tau^*)}{n'(\tau)}(p+1-e_1-e_2)/2 & \text{if } e_1 = e_2. \end{cases}$$

Proposition 6.4 implies that  $n'(\tau)/n'(\tau^*) \in \{1, 2\}$ , and is equal to 1 unless  $e_1 + e_2, e_3, e_4$  are all even. Moreover, if  $n'(\tau) \neq n'(\tau^*)$  then  $\gamma(\tau^*)/\gamma(\tau) = 2$ . The statement of the corollary follows from this.  $\square$

*Remark 6.6.* Similar to the proof of Corollary 6.5, one may show that every genus-0 three-point cover of type  $(p; e_1, e_2, e_3)$  has bad reduction. We do not include this proof here, as a proof of this result using linear series already occurs in [14], Theorem 4.2.

## 7. REDUCTION OF ADMISSIBLE COVERS

In this section, we return to the case of non-Galois covers, and use the results of Section 6 to compute the number of “admissible covers with good reduction”. We start by defining what we mean by this. As always, we fix a type  $(p; e_1, e_2, e_3, e_4)$  with  $1 < e_1 \leq e_2 \leq e_3 \leq e_4 < p$  satisfying the genus-0 condition  $\sum_i e_i = 2p + 2$ . As in Section 2, we consider admissible degenerations of type  $(p; e_1, e_2, *, e_3, e_4)$ , which means that  $Q_3 = \lambda \equiv Q_4 = \infty \pmod{p}$ . Recall from Section 2 that in positive characteristic not every smooth cover degenerates to an admissible cover, as a degeneration might become inseparable. The number of admissible covers (even counted with multiplicity) is still bounded by the number of smooth covers, but equality need not hold.

**Definition 7.1.** We define  $h_p^{\text{adm}}(p; e_1, e_2, *, e_3, e_4)$  as the number of admissible covers of type  $(p; e_1, e_2, *, e_3, e_4)$ , counted with multiplicity, over an algebraically closed field of characteristic  $p$ .

The following proposition is the main result of this section.

**Proposition 7.2.** *The assumptions on the type  $\tau = (p; e_1, e_2, e_3, e_4)$  are as above. Then*

$$h_p^{\text{adm}}(p; e_1, e_2, *, e_3, e_4) > h(p; e_1, e_2, e_3, e_4) - 2p,$$

and

$$h_p^{\text{adm}}(p; e_1, e_2, *, e_3, e_4) = h(p; e_1, e_2, e_3, e_4) - p$$

unless  $e_1 + e_2$  and  $e_3$  are both even.

*Proof.* We begin by noting that in the case  $\tau = (5; 2, 2, 4, 4)$  corresponding to the exceptional case of Corollary 3.2, the assertion of the proposition is automatic since  $h(5; 2, 2, 4, 4) = 8 < 10$ . We may therefore assume that  $\tau \neq (5; 2, 2, 4, 4)$ .

We use the description of the admissible covers in characteristic zero (Theorem 2.3) and the results of Section 6 to estimate the number of admissible covers with good reduction to characteristic  $p$ , i.e. that remain separable.

We first consider the pure-cycle case, i.e. the case of Theorem 2.3.(a). Let  $m$  be an integer satisfying the conditions of *loc. cit.* We write  $f_0 : V_0 \rightarrow X_0$  for the corresponding admissible cover. Recall from Section 2 that  $\bar{X}$  consists of two projective lines  $X_0^1, X_0^2$  intersecting in one point. Choose an irreducible component  $Y_0^i$  of  $V_0$  above  $X_0^i$ , and write  $f_0^i : Y_0^i \rightarrow X_0^i$  for the restriction. These are covers of type  $(d_1; e_1, e_2, m)$  and  $(d_2; m, e_3, e_4)$  with  $d_i \leq p$ , respectively. The admissible cover  $f_0$  has good reduction to characteristic  $p$  if and only if both three-point covers  $f_0^i$  have good reduction.

It is shown in [14], Theorem 4.2, that a genus-0 three-point cover of type  $(d; a, b, c)$  with  $a, b, c < p$  has good reduction to characteristic  $p$  if and only if its degree  $d$  is strictly less than  $p$ . Since the degree  $d_2$  of the cover  $f_0^2$  is always at least as large as the other degree  $d_1$ , it is enough to calculate when  $d_2 < p$ . The

Riemann–Hurwitz formula implies that  $d_2 = (m + e_3 + e_4 - 1)/2$ . Therefore the condition  $d_2 < p$  is equivalent to the inequality

$$e_3 + e_4 + m \leq 2p - 1.$$

Since we assumed the existence of an admissible cover with  $\rho$  an  $m$ -cycle, it follows from Theorem 2.3.(a) that  $m \leq 2d + 1 - e_3 - e_4 = 2p + 1 - e_3 - e_4$ . We find that  $d_2 < p$  unless  $m = 2p + 1 - e_3 - e_4$ . We also note that the lower bound for  $m$  is always less than or equal to the upper bound, which is  $2p + 1 - e_3 - e_4$ . We thus conclude that there are  $2p + 1 - e_3 - e_4$  admissible covers with bad reduction.

We now consider the case of an admissible cover with  $\rho$  an  $e_1$ - $e_2$ -cycle (Theorem 2.3.(b)). Let  $f_0 : V_0 \rightarrow X_0$  be such an admissible cover in characteristic 0, as above. In particular, the restriction  $f_0^1$  (resp.  $f_0^2$ ) has type  $(d_1; e_1, e_2, e_1 - e_2)$  (resp.  $(d_2; e_1 - e_2, e_3, e_4)$ ). We write  $g_0$  for the Galois closure of  $f_0$ , and  $g_0^i$  for the corresponding restrictions. Let  $G^i$  be the Galois group of  $g_0^i$ . The assumptions on the  $e_i$  imply that  $p$  does not divide the order of Galois group of  $g_0^1$ , therefore  $g_0^1$  has good reduction to characteristic  $p$ . Moreover, the cover  $g_0^1$  is uniquely determined by the triple  $(\rho^{-1}, g_3, g_4)$ . If  $e_1 \neq e_2$ , the gluing is likewise uniquely determined, while if  $e_1 = e_2$  there are exactly 2 possibilities for the tuple  $(g_1, g_2, g_3, g_4)$  for a given triple  $(\rho^{-1}, g_3, g_4)$ . Therefore to count the number of admissible covers with bad reduction in this case, it suffices to consider the reduction behavior of the cover  $g_0^2 : Y_0^2 \rightarrow X_0^2$ .

Corollary 6.5 implies that whether or not  $e_1$  equals  $e_2$ , the number of admissible covers with bad reduction in the 2-cycle case is equal to  $(p + 1 - e_1 - e_2)$  unless  $e_1 + e_2$  and  $e_3$  are both even, and bounded from above by  $2(p + 1 - e_1 - e_2)$  always. We conclude using Theorem 2.3 that the total number of admissible covers with bad reduction counted with multiplicity is less than or equal to

$$(2p + 1 - e_3 - e_4) + 2(p + 1 - e_1 - e_2) = p + (p + 1 - e_1 - e_2) < 2p,$$

and equal to

$$(2p + 1 - e_3 - e_4) + (p + 1 - e_1 - e_2) = p$$

unless  $e_1 + e_2$  and  $e_3$  are both even. The proposition follows.  $\square$

*Remark 7.3.* Theorem 4.2 of [14] does not need the assumption  $d = p$ . Therefore the proof of Proposition 7.2 in the single-cycle case shows the following stronger result. Let  $(d; e_1, e_2, e_3, e_4)$  be a genus-0 type with  $1 < e_1 \leq e_2 \leq e_3 \leq e_4 < p$ . Then the number of admissible covers with a single ramified point over the node and bad reduction to characteristic  $p$  is

$$(d - p + 1)(d + p + 1 - e_3 - e_4)$$

when either  $d + 1 \geq e_2 + e_3$  or  $d + 1 - e_1 < p$ . Otherwise, all admissible covers have bad reduction.

## 8. PROOF OF THE MAIN RESULT

In this section, we count the number of mere covers with ramification type  $(p; e_1, e_2, e_3, e_4)$  and bad reduction in the case that the branch points are generic. Equivalently, we compute the  $p$ -Hurwitz number  $h_p(p; e_1, e_2, e_3, e_4)$ .

Suppose that  $r = 4$  and fix a genus-0 type  $\tau = (p; e_1, e_2, e_3, e_4)$  with  $2 \leq e_1 \leq e_2 \leq e_3 \leq e_4 < p$ . We let  $g : Y \rightarrow X = \mathbb{P}_K^1$  be a Galois cover of type  $\tau$  defined over a local field  $K$  as in Section 4, such that  $(X; Q_i)$  is the generic  $r$ -marked curve of



genus 0. It is no restriction to suppose that  $Q_1 = 0, Q_2 = 1, Q_3 = \lambda, Q_4 = \infty$ , where  $\lambda$  is transcendental over  $\mathbb{Q}_p$ . We suppose that  $g$  has bad reduction to characteristic  $p$ , and denote by  $\bar{g} : \bar{Y} \rightarrow \bar{X}$  the stable reduction. We have seen in Section 4 that we may associate with  $\bar{g}$  a set of primitive tail covers  $(\bar{g}_i)$  and a deformation datum  $(\bar{Z}_0, \omega)$ . The primitive tail covers  $\bar{g}_i$  for  $i \in \mathbb{B} = \{1, 2, 3, 4\}$  are uniquely determined by the  $e_i$  (Lemma 5.1).

The following proposition shows that the number of covers with bad reduction is divisible by  $p$  in the case that the branch point are generic.

**Proposition 8.1.** *Suppose that  $(X = \mathbb{P}_K^1; Q_i)$  is the generic  $r = 4$ -marked curve of genus zero. Then the number of mere covers of  $X$  of ramification type  $(p; e_1, e_2, e_3, e_4)$  with bad reduction is nonzero and divisible by  $p$ .*

*Proof.* Since the number of Galois covers and the number of mere covers differ by a prime-to- $p$  factor, it suffices to prove the proposition for Galois covers. The existence portion of the proposition is proved in [3], Proposition 2.4.1, and the divisibility by  $p$  in Lemma 3.4.1 of *loc. cit.* (in a more general setting). We briefly sketch the proof, which is easier in our case due to the simple structure of the stable reduction (Lemma 4.2). The idea of the proof is inspired by a result of [19], Section 3.

We begin by observing that away from the wild branch point  $\xi_i$ , the primitive tail cover  $\bar{g}_i$  is tamely ramified. Therefore we can lift this cover of affine curves to characteristic zero.

Let  $\mathcal{X}_0 = \mathbb{P}_R^1$  be equipped with 4 sections  $Q_1 = 0, Q_2 = 1, Q_3 = \lambda, Q_4 = \infty$ , where  $\lambda \in R$  is transcendental over  $\mathbb{Z}_p$ . Then (4.3) defines an  $m$ -cyclic cover  $\mathcal{Z}_0 \rightarrow \mathcal{X}_0$ . We write  $Z \rightarrow X$  for its generic fiber. Proposition 4.4 implies the existence of a deformation datum  $(\bar{Z}_0, \omega)$ . Associated with the deformation datum is a character  $\chi : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{F}_p^\times$  defined by  $\chi(\beta) = \beta^* z / z \pmod{z}$ . The differential form  $\omega$  corresponds to a  $p$ -torsion point  $P_0 \in J(\bar{Z}_0)[p]_\chi$  on the Jacobian of  $\bar{Z}_0$ . See for example [18] (Here we use that the conjugacy classes  $C_i$  are conjugacy classes of prime-to- $p$  elements. This implies that the differential form  $\omega$  is holomorphic.)

Since  $\sum_{i=1}^4 h_i = 2m$  and the branch points are generic, we have that  $J(\bar{Z}_0)[p]_\chi \simeq \mathbb{Z}/p\mathbb{Z} \times \mu_p$  ([4], Proposition 2.9) After enlarging the discretely valued field  $K$ , if necessary, we may choose a  $p$ -torsion point  $P \in J(\mathcal{Z}_0 \otimes_R K)[p]_\chi$  lifting  $P_0$ . It corresponds to an étale  $p$ -cyclic cover  $W \rightarrow Z$ . The cover  $\psi : W \rightarrow X$  is Galois, with Galois group  $N := \mathbb{Z}/p\mathbb{Z} \rtimes_\chi \mathbb{Z}/m\mathbb{Z}$ . It is easy to see that  $\psi$  has bad reduction, and that its deformation datum is  $(\bar{Z}_0, \omega)$ .

By using formal patching ([16] or [20]), one now checks that there exists a map  $g_R : \mathcal{Y} \rightarrow \mathcal{X}$  of stable curves over  $\text{Spec}(R)$  whose generic fiber is a  $G$ -Galois cover of smooth curves, and whose special fiber defines the given tails covers and the deformation datum. Over a neighborhood of the original component  $g_R$  is the induced cover  $\text{Ind}_N^G \mathcal{Z}_0 \rightarrow \mathcal{X}_0$ . Over the tails, the cover  $g_R$  is induced by the lift of the tail covers. The fact that we can patch the tail covers with the cover over  $\mathcal{X}_0$  follows from the observation that  $h_i < m_i$  (Lemma 5.1), since locally there a unique cover with this ramification ([20], Lemma 2.12). This proves the existence statement.

The divisibility by  $p$  now follows from the observation that the set of lifts  $P$  of the  $p$ -torsion point  $P_0 \in J(\bar{Z}_0)[p]_\chi$  corresponding to the deformation datum is a  $\mu_p$ -torsor.  $\square$

We are now ready to prove our Theorem 1.1, as well as a slightly sharper version of Theorem 1.3.

**Theorem 8.2.** *Let  $p$  be an odd prime and  $k$  an algebraically closed field of characteristic  $p$ . Suppose we are given integers  $2 \leq e_1 \leq e_2 \leq e_3 \leq e_4 < p$ . There exists a dense open subset  $U \subset \mathbb{P}_k^1$  such that for  $\lambda \in U$  the number of degree- $p$  covers with ramification type  $(e_1, e_2, e_3, e_4)$  over the branch points  $(0, 1, \lambda, \infty)$  is given by the formula*

$$h_p(e_1, \dots, e_4) = \min_i (e_i(p + 1 - e_i)) - p.$$

*Furthermore, unless both  $e_1 + e_2$  and  $e_3$  are even, every such cover has good degeneration under a degeneration of the base sending  $\lambda$  to  $\infty$ .*

*Proof.* Proposition 8.1 implies that the number of covers with ramification type  $(p; e_1, e_2, e_3, e_4)$  and bad reduction is at least  $p$ . This implies that the generic Hurwitz number  $h_p(e_1, \dots, e_4)$  is at most  $\min_i (e_i(p + 1 - e_i)) - p$ . Proposition 7.2 implies that the number of admissible covers in characteristic  $p$  strictly larger than  $\min_i (e_i(p + 1 - e_i)) - 2p$ . Since the number of separable covers can only decrease under specialization, we conclude that the generic Hurwitz number equals  $\min_i (e_i(p + 1 - e_i)) - p$ . This proves the first statement, and the second follows immediately from Proposition 7.2 in the situation that  $e_1 + e_2$  and  $e_3$  are not both even.  $\square$

*Remark 8.3.* By using the results of [3] one can prove a stronger result than Theorem 8.2. We say that a  $\lambda \in \mathbb{P}_k^1 \setminus \{0, 1, \infty\}$  is *supersingular* if it is a zero of the polynomial (4.5) and *ordinary* otherwise. Then the number of covers in characteristic  $p$  of type  $(p; e_1, e_2, e_3, e_4)$  branched at  $(0, 1, \lambda, \infty)$  is  $h_p(p; e_1, e_2, e_3, e_4)$  if  $\lambda$  is ordinary and  $h_p(p; e_1, e_2, e_3, e_4) - 1$  if  $\lambda$  is supersingular. To prove this result, one needs to study the stable reduction of the cover  $\pi : \bar{\mathcal{H}} \rightarrow \mathbb{P}_\lambda^1$  of the Hurwitz curve to the configuration space. We do not prove this result here, as it would require too many technical details.

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